Corrigendum to “Determining a sound-soft polyhedral scatterer by a single far-field measurement”

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In the paper, [1], on the determination of a sound-soft polyhedral scatterer by a single far-field measurement, the proof of Proposition 3.2 is incomplete. In this corrigendum we provide a new proof of the same proposition which fills the previous gap. In order to introduce it, we recall some definitions from [1].

Let v be a nontrivial real valued solution to the Helmholtz equation

$$\Delta v + k^2 v = 0 \text{ in } G, \tag{1}$$

in a connected open set $G \subset \mathbb{R}^N$, $N \geq 2$. We denote the *nodal set* of v as

$$\mathcal{N}_v = \{x \in G : \ v(x) = 0\}$$

and we let $\mathcal{C}_v$ be the set of *nodal critical points*, that is

$$\mathcal{C}_v = \{x \in G : \ v(x) = 0 \text{ and } \nabla v(x) = 0\}.$$

We say that $\Sigma \subset \mathcal{N}_v$ is a *regular portion* of $\mathcal{N}_v$ if it is an analytic open and connected hypersurface contained in $\mathcal{N}_v \backslash \mathcal{C}_v$. Let us denote by $A_1, A_2, \ldots, A_n, \ldots$ the *nodal domains* of v in G, that is the connected components of $\{x \in G : v(x) \neq 0\} = G \backslash \mathcal{N}_v$. Let us recall the statement of Proposition 3.2 in [1].

Proposition 3.2 ([1]) *We can order the nodal domains $A_1, A_2, \ldots, A_n, \ldots$ in such a way that for any $j \geq 2$ there exist i, $1 \leq i < j$, and a regular portion Σ_j of $\mathcal{N}_v$ such that*

$$\Sigma_j \subset \partial A_i \cap \partial A_j. \tag{2}$$

The gap in the proof given in [1] stands in the fact that the ordering $A_1, A_2, \ldots, A_n, \ldots$ obtained with that method might not ensure that all the nodal domains are contained in the sequence. We base the new proof on the following theorem.

Theorem 1 *The set $\mathcal{C}_v$ has Hausdorff dimension not exceeding $N - 2$.*

A proof can be found in [5, Theorem 2.1]. Further developments of the theory on the structure of zero sets of solutions to elliptic equations can be found, for instance, in [2, 3] and in their references.

Let $G' = G\backslash\mathcal{C}_v$. By the property of $\mathcal{C}_v$ described in the previous theorem, and by using [4, Chapter VII, Section 4] and [4, Theorem IV 4, Corollary 2], we can conclude that G' is an open and connected set. We also remark that, for every $x \in \mathcal{N}_v\backslash\mathcal{C}_v$, there are exactly two nodal domains, A and B, of v such that $x \in \partial A \cap \partial B$. Finally, let us note that the nodal domains of v in G coincide with the nodal domains of v in G'.

We shall also make use of the following elementary lemma.

Lemma 2 *For any connected open set $G \subset \mathbb{R}^N$, there exists an increasing sequence $\{G_m\}_{m=1}^{\infty}$ of bounded, connected open sets such that $G = \bigcup_{m=1}^{\infty} G_m$ and $G_m \subset\subset G$ for every m.*

Proof. For every $k = 1, 2, \ldots$, we denote

$$D_k = \{x \in G : \ \mathrm{dist}(x, \partial G) > 1/k, \ |x| < k\}.$$

Let us assume, without loss of generality, that $D_1 \neq \emptyset$ and let us fix $y \in D_1$. For every $x \in \overline{D_k}$, let γ_x be a path in G joining y to x. For every $h > 0$, let $\mathcal{U}_x^h = \{z \in \mathbb{R}^N : \ \mathrm{dist}(z, \gamma_x) < h\}$. We obviously have that $\mathcal{U}_x^h$ is a connected open set. Let $h(x) > 0$ be such that $\mathcal{U}_x^{h(x)} \subset\subset G$. We have that $\{\mathcal{U}_x^{h(x)}\}_{x\in\overline{D_k}}$ is an open covering of the compact set $\overline{D_k}$. Therefore, we can find $x_1, \ldots, x_l \in \overline{D_k}$ such that $\overline{D_k} \subset \bigcup_{j=1}^{l} \mathcal{U}_{x_j}^{h(x_j)}$. We observe that $E_k = \bigcup_{j=1}^{l} \mathcal{U}_{x_j}^{h(x_j)}$ is an open connected set such that $\overline{D_k} \subset E_k \subset\subset G$. Therefore the lemma follows choosing $G_m = \bigcup_{k=1}^{m} E_k$. □

Proof of Proposition 3.2. We apply Lemma 2 to the connected set $G' = G\backslash\mathcal{C}_v$. We choose A_1 such that $A_1 \cap G_1 \neq \emptyset$ and we proceed by induction.

Let us assume that we have ordered $A_1, \ldots, A_n$ in such a way that there exist $\Sigma_2, \ldots, \Sigma_n$ regular portions of $\mathcal{N}_v$ such that (2) holds for any $j = 2, \ldots, n$ and for some $i < j$.

Let $\hat{A}_n = \overset{\circ}{\overline{A_1 \cup \ldots \cup A_n}}$. If $G'\backslash\hat{A}_n = \emptyset$, then we are done. Otherwise, let $m \geq 1$ be the smallest number such that $G_m\backslash\hat{A}_n \neq \emptyset$. Since G_m is connected, we can find $y \in \partial\hat{A}_n \cap G_m$ and $r > 0$ such that $B_r(y) \cap \partial\hat{A}_n$ is a regular portion of $\mathcal{N}_v$ and there exist exactly two nodal domains, $\tilde{A}_1 \subset \hat{A}_n$ and $\tilde{A}_2$ with $\tilde{A}_2 \cap \hat{A}_n = \emptyset$, whose intersections with $B_r(y)$ are not empty. Clearly, $\tilde{A}_1$ coincides with A_i, for some $i = 1, \ldots, n$, and if we pick $A_{n+1} = \tilde{A}_2$ and $\Sigma_{n+1} = B_r(y) \cap \mathcal{N}_v$, then (2) holds for $j = n + 1$, too.

If G contains only finitely many nodal domains, then we can iterate this construction and after a finite number of steps we recover all the nodal domains, that is for some $l \in \mathbb{N}$ we have $G'\backslash\hat{A}_l = \emptyset$ and we are done. Otherwise, we argue in the following way. Since $\overline{G_m}$ is contained in G', for every $x \in \overline{G_m}$ there is a neighbourhood of x intersecting at most two different nodal domains. By compactness, we obtain that $\overline{G_m}$ intersects at most finitely many different nodal domains. Hence, if we iterate the previous construction, after a finite number of steps we find $l \in \mathbb{N}$ such that $G_m\backslash\hat{A}_l = \emptyset$. By repeating the argument for the smallest $m' > m$ such that $G_{m'}\backslash\hat{A}_l \neq \emptyset$, we conclude that for any $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $G_m\backslash\hat{A}_l = \emptyset$. Therefore the infinite sequence $\{A_i\}$ comprises all the nodal domains of v in G. □

Acknowledgements

The authors wish to express their gratitude to Hongyu Liu and Jun Zou for pointing out to them the gap in the proof of Proposition 3.2 in [1] and for kindly sending them their preprint [6].

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